

# Upper bound on list-decoding radius of binary codes

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**Abstract**—Consider the problem of packing Hamming balls of a given relative radius subject to the constraint that they cover any point of the ambient Hamming space with multiplicity at most  $L$ . For odd  $L \geq 3$  an asymptotic upper bound on the rate of any such packing is proven. The resulting bound improves the best known bound (due to Blinovsky’1986) for rates below a certain threshold. The method is a superposition of the linear-programming idea of Ashikhmin, Barg and Litsyn (that was used previously to improve the estimates of Blinovsky for  $L = 2$ ) and a Ramsey-theoretic technique of Blinovsky. As an application it is shown that for all odd  $L$  the slope of the rate-radius tradeoff is zero at zero rate.

**Index Terms**—Combinatorial coding theory, list-decoding, converse bounds

## I. MAIN RESULT AND DISCUSSION

One of the most well-studied problems in information theory asks to find the maximal rate at which codewords can be packed in binary space with a given minimum distance between codewords. Operationally, this (still unknown) rate gives the capacity of the binary input-output channel subject to adversarial noise of a given level. A natural generalization was considered by Elias and Wozencraft [1], [2], who allowed the decoder to output a list of size  $L$ . In this paper we provide improved upper bounds on the latter question.

Our interest in bounding the asymptotic tradeoff for the list-decoding problem is motivated by our study of fundamental limits of joint source-channel communication [3]. Namely, in [4, Theorem 6] we proposed an extension of the previous result in [3, Theorem 7] that required bounding rate for the list-decoding problem.

We proceed to formal definitions and brief overview of known results. For a binary code  $\mathcal{C} \subset \mathbb{F}_2^n$  we define its list-size  $L$  decoding radius as

$$\tau_L(\mathcal{C}) \triangleq \frac{1}{n} \max\{r : \forall x \in \mathbb{F}_2^n \mid |\mathcal{C} \cap \{x + B_r^n\}| \leq L\},$$

where Hamming ball  $B_r^n$  and Hamming sphere  $S_r^n$  are defined as

$$B_r^n \triangleq \{x \in \mathbb{F}_2^n : |x| \leq r\}, \quad (1)$$

$$S_r^n \triangleq \{x \in \mathbb{F}_2^n : |x| = r\} \quad (2)$$

with  $|x| = |\{i : x_i = 1\}|$  denoting the Hamming weight of  $x$ . Alternatively, we may define  $\tau_L$  as follows:<sup>1</sup>

$$\tau_L(\mathcal{C}) = \frac{1}{n} \left( \min \left\{ \text{rad}(S) : S \in \binom{\mathcal{C}}{L+1} \right\} - 1 \right),$$

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<sup>1</sup> $\binom{\mathcal{C}}{j}$  denotes the set of all subsets of  $\mathcal{C}$  of size  $j$ .

where  $\text{rad}(S)$  denotes radius of the smallest ball containing  $S$  (known as Chebyshev radius):

$$\text{rad}(S) \triangleq \min_{y \in \mathbb{F}_2^n} \max_{x \in S} |y - x|.$$

The asymptotic tradeoff between rate and list-decoding radius  $\tau_L$  is defined as usual:

$$\tau_L^*(R) \triangleq \limsup_{n \rightarrow \infty} \max_{\mathcal{C} : |\mathcal{C}| \geq 2^{nR}} \tau_L(\mathcal{C}) \quad (3)$$

$$R_L^*(\tau) \triangleq \limsup_{n \rightarrow \infty} \max_{\mathcal{C} : \tau_L(\mathcal{C}) \geq \tau} \frac{1}{n} \log |\mathcal{C}| \quad (4)$$

The best known upper (converse) bounds on this tradeoff are as follows:

- List size  $L = 1$ : The best bound to date was found by McEliece, Rodemich, Rumsey and Welch [5]:

$$R_1^*(\tau) \leq R_{LP2}(2\tau), \quad (5)$$

$$R_{LP2}(\delta) \triangleq \min \log 2 - h(\alpha) + h(\beta), \quad (6)$$

where  $h(x) = -x \log x - (1-x) \log(1-x)$  and minimum is taken over all  $0 \leq \beta \leq \alpha \leq 1/2$  satisfying

$$2 \frac{\alpha(1-\alpha) - \beta(1-\beta)}{1 + 2\sqrt{\beta(1-\beta)}} \leq \delta$$

For rates  $R < 0.305$  this bound coincides with the simpler bound:

$$\tau_1^*(R) \leq \frac{1}{2} \delta_{LP1}(R), \quad (7)$$

$$\delta_{LP1}(R) \triangleq \frac{1}{2} - \sqrt{\beta(1-\beta)}, \quad R = \log 2 - h(\beta), \quad (8)$$

where  $\beta \in [0, \frac{1}{2}]$ .

- List size  $L = 2$ : The bound found by Ashikhmin, Barg and Litsyn [6] is given as<sup>2</sup>

$$R_2^*(\tau) \leq \log 2 - h(2\tau) + R_{up}(2\tau, 2\tau),$$

where  $R_{up}(\delta, \alpha)$  is the best known upper bound on rate of codes with minimal distance  $\delta n$  constrained to live on Hamming spheres  $S_{\alpha n}^n$ . The expression for  $R_{up}(\delta, \alpha)$  can be obtained by using the linear programming bound from [5] and applying Levenshtein’s monotonicity, cf. [7, Lemma 4.2(6)]. The resulting expression is

$$R_2^*(\tau) \leq \begin{cases} R_{LP2}(2\tau), & \tau \leq \tau_0 \\ \log 2 - h(2\tau) + h(u(\tau)), & \tau > \tau_0, \end{cases} \quad (9)$$

where  $\tau_0 \approx 0.1093$  and

$$u(\tau) = \frac{1}{2} - \sqrt{\frac{1}{4} - (\sqrt{\tau - 3\tau^2} - \tau)^2}$$

<sup>2</sup>This result follows from optimizing [6, Theorem 4]. It is slightly stronger than what is given in [6, Corollary 5].

(cf. [7, (9)]).

- For list sizes  $L \geq 3$ : The original bound of Blinovsky [8] appears to be the best (before this work):

$$\tau_L^*(R) \leq \sum_{i=1}^{\lceil L/2 \rceil} \frac{\binom{2i-2}{i-1}}{i} (\lambda(1-\lambda))^i, \quad R = 1 - h(\lambda), \quad (10)$$

where  $\lambda \in [0, \frac{1}{2}]$ . Note that [8] also gives a non-constructive lower bound on  $\tau_L^*(R)$ . Results on list-decoding over non-binary alphabets are also known, see [9], [10].

In this paper we improve the bound of Blinovsky for lists of odd size and rates below a certain threshold. To that end we will mix the ideas of Ashikhmin, Barg and Litsyn (namely, extraction of a large spectrum component from the code) and those of Blinovsky (namely, a Ramsey-theoretic reduction to study of symmetric subcodes).

To present our main result, we need to define exponent of Krawtchouk polynomial  $K_{\beta n}(\xi n) = \exp\{nE_\beta(\xi) + o(n)\}$ . For  $\xi \in [0, \frac{1}{2} - \sqrt{\beta(1-\beta)}]$  the value of  $E_\beta(\xi)$  was found in [11]. Here we give it in the following parametric form, cf. [12] or [13, Lemma 4]:

$$E_\beta(\xi) = \xi \log(1-\omega) + (1-\xi) \log(1+\omega) - \beta \log \omega \quad (11)$$

$$\xi = \frac{1}{2}(1 - (1-\beta)\omega - \beta\omega^{-1}), \quad (12)$$

where

$$\omega \in \left[ \frac{\beta}{1-\beta}, \sqrt{\frac{\beta}{1-\beta}} \right].$$

Our main result is the following:

**Theorem 1.** Fix list size  $L \geq 2$ , rate  $R$  and an arbitrary  $\beta \in [0, 1/2]$  with  $h(\beta) \leq R$ . Then any sequence of codes  $\mathcal{C}_n \subset \{0, 1\}^n$  of rate  $R$  satisfies

$$\limsup_{n \rightarrow \infty} \tau_L(\mathcal{C}_n) \leq \max_{j, \xi_0} \xi_0 g_j \left( 1 - \frac{\xi_1}{2\xi_0} \right) + (1 - \xi_0) g_j \left( \frac{\xi_1}{2(1 - \xi_0)} \right), \quad (13)$$

where maximization is over  $\xi_0$  satisfying

$$0 \leq \xi_0 \leq \frac{1}{2} - \sqrt{\beta(1-\beta)} \quad (14)$$

and  $j$  ranging over  $\{0, 1, 3, \dots, 2k+1, \dots, L\}$  if  $L$  is odd and over  $\{0, 2, \dots, 2k, \dots, L\}$  if  $L$  is even. Quantity  $\xi_1 = \xi_1(\xi_0, \delta, R)$  is a unique solution of

$$R + h(\beta) - 2E_\beta(\xi_0) = h(\xi_0) - \xi_0 h \left( \frac{\xi_1}{2\xi_0} \right) - (1 - \xi_0) h \left( \frac{\xi_1}{2(1 - \xi_0)} \right), \quad (15)$$

on the interval  $[0, 2\xi_0(1 - \xi_0)]$  and functions  $g_j(\nu)$  are defined as

$$g_j(\nu) \triangleq \frac{1}{L+j} (L\nu - \mathbb{E}[\lceil 2W - L - j \rceil^+]), \quad W \sim \text{Bino}(L, \nu) \quad (16)$$

As usual with bounds of this type, cf. [14], it appears that taking  $h(\beta) = R$  can be done without loss. Under such choice,

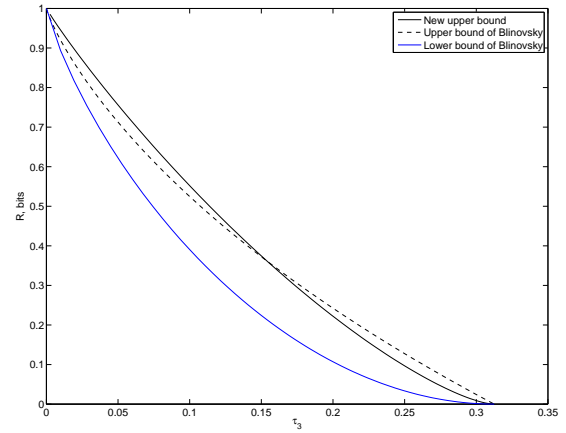


Fig. 1. Comparison of bounds on  $R_L^*(\tau)$  for list size  $L = 3$

TABLE I  
RATES FOR WHICH NEW BOUND\* IMPROVES STATE OF THE ART

List size $L$	Range of rates
$L = 3$	$0 < R \leq 0.361$
$L = 5$	$0 < R \leq 0.248$
$L = 7$	$0 < R \leq 0.184$
$L = 9$	$0 < R \leq 0.136$
$L = 11$	$0 < R \leq 0.100$

\* This is computation of (13) with  $h(\beta) = R$ .

our bound outperforms Blinovsky's for all odd  $L$  and all rates small enough (see Corollary 3 below). The bound for  $L = 3$  is compared in Fig. 1 with the result of Blinovsky numerically. For larger odd  $L$  the comparison is similar, but the range of rates where our bound outperforms Blinovsky's becomes smaller, see Table I.

Evaluation of Theorem 1 is computationally possible, but is somewhat tedious. Fortunately, for small  $L$  the maximum over  $\xi_0$  and  $j$  is attained at  $\xi_0 = \frac{1}{2} - \sqrt{\beta(1-\beta)}$  and  $j = 1$ . We rigorously prove this for  $L = 3$ :<sup>3</sup>

**Corollary 2.** For list-size  $L = 3$  we have

$$\tau_L^*(R) \leq \frac{3}{4}\delta - \frac{1}{16} \left( \frac{(2\delta - \xi_1)^3}{\delta^2} + \frac{\xi_1^3}{(1-\delta)^2} \right), \quad (17)$$

where  $\delta \in (0, 1/2]$  and  $\xi_1 \in [0, 2\delta(1-\delta)]$  are functions of  $R$  determined from

$$R = h \left( \frac{1}{2} - \sqrt{\delta(1-\delta)} \right), \quad (18)$$

$$R = \log 2 - \delta h \left( \frac{\xi_1}{2\delta} \right) - (1-\delta) h \left( \frac{\xi_1}{2(1-\delta)} \right) \quad (19)$$

Another interesting implication of Theorem 1 is that it allows us to settle the question of slope of the curve  $R_L^*(\tau)$  at zero rate. Notice that Blinovsky's converse bound (10) has a negative slope, while his achievability bound has a zero slope. Our bound always has a zero slope for odd  $L$  (but not for even  $L$ , see Remark 2 in Section II-C):

<sup>3</sup>Notice that proofs of each of the two Corollaries below contain different relaxations of the bound (13), e.g. (22), which are easier to evaluate. Notice also that in Table I for the last two entries ( $L = 9, 11$ ) at the high endpoint of rate the maximum over  $\xi_0$  is attained *not* at  $\frac{1}{2} - \sqrt{\beta(1-\beta)}$ .

**Corollary 3.** Fix arbitrary odd  $L \geq 3$ . There exists  $R_0 = R_0(L) > 0$  such that for all rates  $R < R_0$  we have

$$\tau_L^*(R) \leq g_1(\delta_{LP1}(R)), \quad (20)$$

where  $g_1(\cdot)$  is a degree- $L$  polynomial defined in (16). In particular,

$$\left. \frac{d}{d\tau} \right|_{\tau=\tau_L^*(0)} R_L^*(\tau) = 0, \quad (21)$$

where the zero-rate radius is  $\tau_L^*(0) = \frac{1}{2} - 2^{-L-1} \binom{L-1}{\frac{L-1}{2}}$ .

Before closing our discussion we make some additional remarks:

- 1) The bound in Theorem 1 can be slightly improved by replacing  $\delta_{LP1}(R)$ , that appears in the right-hand side of (14), with a better bound, a so-called second linear-programming bound  $\delta_{LP2}(R)$  from [5]. This would enforce the usage of the more advanced estimate of Litsyn [15, Theorem 5] and complicate analysis significantly. Notice that  $\delta_{LP2}(R) \neq \delta_{LP1}(R)$  only for rates  $R \geq 0.305$ . If we focus attention only on rates where new bound is better than Blinovsky's, such a strengthening only affects the case of  $L = 3$  and results in a rather minuscule improvement (for example, for rate  $R = 0.33$  the improvement is  $\approx 3 \cdot 10^{-5}$ ).
- 2) For even  $L$  it appears that  $h(\beta) = R$  is no longer optimal. However, the resulting bound does not appear to improve upon Blinovsky's.
- 3) When  $L$  is large (e.g. 35) the maximum in (13) is not always attained by either  $j = 1$  or  $\xi_0 = \delta_{LP1}(R)$ . It is not clear whether such anomalies only happen in the region of rates where our bound is inferior to Blinovsky's.
- 4) The result of Corollary 3 follows by weakening (13) (via concavity of  $g_j$ , Lemma 8) to

$$\limsup_{n \rightarrow \infty} \tau_L(\mathcal{C}_n) \leq \max_{j, \xi_0} g_j(\xi_0) = \max_j g_j(\delta_{LP1}(R)). \quad (22)$$

The  $R < R_0(L)$  condition is only used to show that the maximum is attained at  $j = 1$ . Note also that weakening (22) corresponds to omitting the extra Elias-Bassalygo type reduction, which is responsible for the extra optimization over  $\xi_1$  in (13).

Finally, at the invitation of anonymous reviewer we give our intuition about why our bound outperforms Blinovsky's for odd  $L$ . It is easiest to compare with the weakening (22) of our bound. Now compare the two proofs:

- 1) Blinovsky [8] first uses Elias-Bassalygo reduction to restrict attention to a subcode  $\mathcal{C}'$  situated on a Hamming sphere of radius  $\approx \delta_{GV}(R) = h^{-1}(1 - R)$ . Then he proves an upper bound for  $\tau_L(\mathcal{C}')$  valid as long as  $|\mathcal{C}'| \gg 1$  via a Plotkin-type argument together with a great symmetrization idea.
- 2) Our bound (following Ashikhmin, Barg and Litsyn [6]) instead uses a Kalai-Linial [11] reduction to select a subcode  $\mathcal{C}''$  situated on a Hamming sphere of radius

$\approx \delta_{LP1}(R)$ . We then proceeded to prove a (Plotkin-type) upper bound on a strange quantity:

$$\tau_L^o(\mathcal{C}'') = \frac{1}{n} \left( \min \left\{ \text{rad}(\{0\} \cup S) : S \in \binom{\mathcal{C}}{L} \right\} - 1 \right),$$

which corresponds to a requirement that the code contain not more than  $L - 1$  codewords in any ball of radius  $\tau_L^o$ , but only for those balls that happen to also contain the origin.

Notice that the sphere returned by Kalai-Linial is bigger than that of Elias-Bassalygo (which is the reason our bound deteriorates at large rates), but the good thing is that the subcode  $\mathcal{C}''$  has another codeword  $c_0$  at the center of the Hamming sphere. Now, intuitively  $\tau_L^o$  is roughly equivalent to  $\tau_{L-1}$ . The zero-rate (Plotkin) radius for a list- $L$  decoding of binary codes on Hamming sphere  $S_{\xi_n}^n$  is given by

$$p_L(\xi) = \frac{\mathbb{E}[\min(W_\xi, L + 1 - W_\xi)]}{L + 1}, \quad W_\xi \sim \text{Bino}(L + 1, \xi).$$

So intuitively, we expect that Blinovsky's bound should give

$$\tau_L^*(R) \lesssim p_L(\delta_{GV}(R))$$

while our bound should give

$$\tau_L^*(R) \lesssim p_{L-1}(\delta_{LP1}(R)).$$

Finally, it is easy to check that for even  $L$  we have  $p_L = p_{L-1}$ , while for odd  $L$ ,  $p_L > p_{L-1}$ . This is the main intuitive reason why our bound succeeds in improving Blinovsky's, but only for odd  $L$ .

## II. PROOFS

### A. Proof of Theorem 1

Consider an arbitrary sequence of codes  $\mathcal{C}_n$  of rate  $R$ . As in [6] we start by using Delsarte's linear programming to select a large component of the distance distribution of the code. Namely, we apply result of Kalai and Linial [11, Proposition 3.2]: For every  $\beta$  with  $h(\beta) \leq R$  there exists a sequence  $\epsilon_n \rightarrow 0$  such that for every code  $\mathcal{C}$  of rate  $R$  there is a  $\xi_0$  satisfying (14) such that

$$\begin{aligned} A_{\xi_0 n}(\mathcal{C}) &\triangleq \frac{1}{|\mathcal{C}|} \sum_{x, x' \in \mathcal{C}} 1\{|x - x'| = \xi_0 n\} \\ &\geq \exp\{n(R + h(\beta) - 2E_\beta(\xi_0) + \epsilon_n)\}. \end{aligned} \quad (23)$$

Without loss of generality (by compactness of the interval  $[0, 1/2 - \sqrt{\beta(1 - \beta)}]$  and passing to a proper subsequence of codes  $\mathcal{C}_{n_k}$ ) we may assume that  $\xi_0$  selected in (23) is the same for all blocklengths  $n$ . Then there is a sequence of subcodes  $\mathcal{C}'_n$  of asymptotic rate

$$R' \geq R + h(\beta) - 2E_\beta(\xi_0)$$

such that each  $\mathcal{C}'_n$  is situated on a sphere  $c_0 + S_{\xi_0}$  surrounding another codeword  $c_0 \in \mathcal{C}$ . Our key geometric result is: If there are too many codewords on a sphere  $c_0 + S_{\xi_0}$  then it is possible to find  $L$  of them that are includable in a small ball that also contains  $c_0$ . Precisely, we have:

**Lemma 4.** Fix  $\xi_0 \in (0, 1)$  and positive integer  $L$ . There exist a sequence  $\epsilon_n \rightarrow 0$  such that for any code  $\mathcal{C}'_n \subset \mathcal{S}_{\xi_0 n}$  of rate  $R' > 0$  there exist  $L$  codewords  $c_1, \dots, c_L \in \mathcal{C}'_n$  such that

$$\frac{1}{n} \text{rad}(0, c_1, \dots, c_L) \leq \theta(\xi_0, R', L) + \epsilon_n, \quad (24)$$

where

$$\theta(\xi_0, R', L) \triangleq \max_j \theta_j(\xi_0, R', L) \quad (25)$$

$$\theta_j(\xi_0, R', L) \triangleq \xi_0 g_j \left( 1 - \frac{\xi_1}{2\xi_0} \right) + (1 - \xi_0) g_j \left( \frac{\xi_1}{2(1 - \xi_0)} \right), \quad (26)$$

with  $\xi_1 = \xi_1(\xi_0)$  found as unique solution on interval  $[0, 2\xi_0(1 - \xi_0)]$  of

$$R' = h(\xi_0) - \xi_0 h \left( \frac{\xi_1}{2\xi_0} \right) - (1 - \xi_0) h \left( \frac{\xi_1}{2(1 - \xi_0)} \right), \quad (27)$$

functions  $g_j$  are defined in (16) and  $j$  in maximization (25) ranging over the same set as in Theorem 1.

Equipped with Lemma 4 we immediately conclude that

$$\limsup_{n \rightarrow \infty} \tau_L(\mathcal{C}_n) \leq \max_{\xi_0 \in [0, \delta]} \theta(\xi_0, R + h(\beta) - 2E_\beta(\xi_0), L). \quad (28)$$

Clearly, (28) coincides with (13). So it suffices to prove Lemma 4.

#### B. Proof of Lemma 4

Let  $\mathcal{T}_L$  be the  $(2^L - 1)$ -dimensional space of probability distributions on  $\mathbb{F}_2^L$ . If  $T \in \mathcal{T}_L$  then we have

$$T = (t_v, v \in \mathbb{F}_2^L) \quad t_v \geq 0, \sum_v t_v = 1.$$

We define distance on  $\mathcal{T}_L$  to be the  $L_\infty$  one:

$$\|T - T'\| \triangleq \max_{v \in \mathbb{F}_2^L} |t_v - t'_v|.$$

Permutation group  $S_L$  acts naturally on  $\mathbb{F}_2^L$  and this action descends to probability distributions  $\mathcal{T}_L$ . We will say that  $T$  is symmetric if

$$T = \sigma(T) \iff t_v = t_{\sigma(v)} \quad \forall v \in \mathbb{F}_2^L$$

for any permutation  $\sigma : [L] \rightarrow [L]$ . Note that symmetric  $T$  is completely specified by  $L + 1$  numbers (weights of Hamming spheres in  $\mathbb{F}_2^L$ ):

$$\sum_{v: |v|=j} t_v, \quad j = 0, \dots, L.$$

Next, fix some total ordering of  $\mathbb{F}_2^n$  (for example, lexicographic). Given a subset  $S \subset \mathbb{F}_2^n$  we will say that  $S$  is given in ordered form if  $S = \{x_1, \dots, x_{|S|}\}$  and  $x_1 < x_2 < \dots < x_{|S|}$  under the fixed ordering on  $\mathbb{F}_2^n$ . For any subset of codewords  $S = \{x_1, \dots, x_L\}$  given in ordered form we define its *joint type*  $T(S)$  as an element of  $\mathcal{T}_L$  with

$$t_v \triangleq \frac{1}{n} |\{j : x_1(j) = v_1, \dots, x_L(j) = v_j\}|,$$

where here and below  $y(j)$  denotes the  $j$ -th coordinate of binary vector  $y \in \mathbb{F}_2^n$ . In this way every subset  $S$  is associated

to an element of  $\mathcal{T}_L$ . Note that  $T(S)$  is symmetric if and only if the  $L \times n$  binary matrix representing  $S$  (by combining row-vectors  $x_j$ ) has the property that the number of columns equal to  $[1, 0, \dots, 0]^T$  is the same as the number of columns  $[0, 1, \dots, 0]^T$  etc. For any code  $\mathcal{C} \subset \mathbb{F}_2^n$  we define its average joint type:

$$\bar{T}_L(\mathcal{C}) = \frac{1}{L! \cdot \binom{[C]}{L}} \sum_{\sigma} \sum_{S \in \binom{\mathcal{C}}{L}} \sigma(T(S)).$$

Evidently,  $\bar{T}_L(\mathcal{C})$  is symmetric.

Our proof crucially depends on a (slight extension of the) brilliant idea of Blinovsky [8]:

**Lemma 5.** For every  $L \geq 1$ ,  $K \geq L$  and  $\delta > 0$  there exist a constant  $K_1 = K_1(L, K, \delta)$  such that for all  $n \geq 1$  and all codes  $\mathcal{C} \subset \mathbb{F}_2^n$  of size  $|\mathcal{C}| \geq K_1$  there exists a subcode  $\mathcal{C}' \subset \mathcal{C}$  of size at least  $K$  such that for any  $S \in \binom{\mathcal{C}'}{L}$  we have

$$\|T(S) - \bar{T}_L(\mathcal{C}')\| \leq \delta. \quad (29)$$

**Remark 1.** Note that if  $S' \subset S$  then every element of  $T(S')$  is a sum of  $\leq 2^L$  elements of  $T(S)$ . Hence, joint types  $T(S')$  are approximately symmetric also for smaller subsets  $|S'| < L$ .

*Proof:* We first will show that for any  $\delta_1 > 0$  and sufficiently large  $|\mathcal{C}|$  we may select a subcode  $\mathcal{C}'$  so that the following holds: For any pair of subsets  $S, S' \subset \mathcal{C}'$  s.t.  $|S| = |S'| \leq L$  we have:

$$\|T(S) - T(S')\| \leq \delta_1 \quad (30)$$

Consider any code  $\mathcal{C}_1 \subset \mathbb{F}_2^n$  and define a hypergraph with vertices indexed by elements of  $\mathcal{C}$  and hyper-edges corresponding to each of the subsets of size  $L$ . Now define a  $\delta_1/2$ -net on the space  $\mathcal{T}_L$  and label each edge according to the closest element of the  $\delta_1/2$ -net. By a theorem of Ramsey there exists  $K_L$  such that if  $|\mathcal{C}_1| \geq K_L$  then there is a subset  $\mathcal{C}'_1 \subset \mathcal{C}$  such that  $|\mathcal{C}'_1| \geq K$  and each of the internal edges, indexed by  $\binom{\mathcal{C}'_1}{L}$ , is assigned the same label. Thus, by triangle inequality (30) follows for all  $S, S' \in \binom{\mathcal{C}'_1}{L}$ .

Next, apply the previous argument to show that there is a constant  $K_{L-1}$  such that for any  $\mathcal{C}_2 \subset \mathbb{F}_2^n$  of size  $|\mathcal{C}_2| \geq K_{L-1}$  there exists a subcode  $\mathcal{C}'_2$  of size  $|\mathcal{C}'_2| \geq K_L$  satisfying (30) for all  $S, S' \in \binom{\mathcal{C}'_2}{L-1}$ . Since  $\mathcal{C}'_2$  satisfies the size assumption on  $\mathcal{C}_1$  made in previous paragraph, we can select a further subcode  $\mathcal{C}''_2 \subset \mathcal{C}'_2$  of size  $\geq K_L$  so that for  $\mathcal{C}''_2$  property (30) holds for all  $S, S'$  of size  $L$  or  $L - 1$ .

Continuing similarly, we may select a subcode  $\mathcal{C}'$  of arbitrary  $\mathcal{C}$  such that (30) holds for all  $|S| = |S'| \leq L$  provided that  $|\mathcal{C}| \geq K_1$ .

Next, we show that (30) implies

$$\|T(S_0) - \sigma(T(S_0))\| \leq C\delta_1, \quad (31)$$

where  $S_0 \in \binom{\mathcal{C}'}{L}$  is arbitrary and  $C = C(L)$  is a constant depending on  $L$  only.

Now to prove (31) let  $T(S_0) = \{t_v, v \in \mathbb{F}_2^L\}$  and consider an arbitrary transposition  $\sigma : [L] \rightarrow [L]$ . It will be clear that our proof does not depend on what transposition is chosen, so



for simplicity we take  $\sigma = \{(L-1) \leftrightarrow L\}$ . We want to show that (30) implies

$$|t_v - t_{\sigma(v)}| \leq \delta_1. \quad \forall v \in \mathbb{F}_2^L \quad (32)$$

Since transpositions generate permutation group  $S_L$ , (31) then follows. Notice that (32) is only informative for  $v$  whose last two digits are not equal, say  $v = [v_0, 0, 1]$ . Suppose that  $S_0 = \{c_1, \dots, c_L\}$  given in the ordered form. Let

$$S = \{c_1, \dots, c_{L-1}\}, \quad (33)$$

$$S' = \{c_1, \dots, c_{L-2}, c_L\} \quad (34)$$

Joint types  $T(S)$  and  $T(S')$  are expressible as functions of  $T(S_0)$  in particular, the number of occurrences of element  $[v_0, 0]$  in  $S$  is  $t_{[v_0, 0, 1]} + t_{[v_0, 0, 0]}$  and in  $S'$  is  $t_{[v_0, 0, 0]} + t_{[v_0, 1, 0]}$ . Thus, from (30) we obtain:

$$|(t_{[v_0, 0, 1]} + t_{[v_0, 0, 0]}) - (t_{[v_0, 0, 0]} + t_{[v_0, 1, 0]})| \leq \delta$$

implying (32) and thus (31).

Finally, we show that (31) implies (29). Indeed, consider the chain

$$\begin{aligned} & \|T(S) - \bar{T}_L(C')\| \\ &= \left\| T(S) - \frac{1}{L! \cdot \binom{|C'|}{L}} \sum_{\sigma} \sum_{S' \in \binom{C'}{L}} \sigma(T(S')) \right\| \end{aligned} \quad (35)$$

$$\leq \frac{1}{L! \cdot \binom{|C'|}{L}} \sum_{\sigma} \sum_{S' \in \binom{C'}{L}} \|T(S) - \sigma(T(S'))\| \quad (36)$$

$$\begin{aligned} & \leq \frac{1}{L! \cdot \binom{|C'|}{L}} \sum_{\sigma} \sum_{S' \in \binom{C'}{L}} \|T(S) - T(S')\| \\ & + \|T(S') - \sigma(T(S'))\| \end{aligned} \quad (37)$$

$$\leq (1 + C)\delta_1, \quad (38)$$

where (36) is by convexity of the norm, (37) is by triangle inequality and (38) is by (30) and (31). Consequently, setting  $\delta_1 = \frac{\delta}{1+C}$  we have shown (29). ■

Before proceeding further we need to define the concept of an average radius (or a moment of inertia):

$$\overline{\text{rad}}(x_1, \dots, x_m) \triangleq \min_y \frac{1}{m} \sum_{i=1}^m |x_i - y|.$$

Note that the minimizing  $y$  can be computed via a per-coordinate majority vote (with arbitrary tie-breaking for even  $m$ ). Consider now an arbitrary subset  $S = \{c_1, \dots, c_L\}$  and define for each  $j \geq 0$  the following functions

$$h_j(S) \triangleq \frac{1}{n} \overline{\text{rad}}(\underbrace{0, \dots, 0}_{j \text{ times}}, c_1, \dots, c_L).$$

It is easy to find an expression for  $h_j(S)$  in terms of the joint-type of  $S$ :

$$h_j(S) = \frac{1}{L+j} (\mathbb{E}[W] - \mathbb{E}[|2W - L - j|^+]) \quad (39)$$

$$\mathbb{P}[W = w] = \sum_{v: |v|=w} t_v, \quad (40)$$

where  $t_v$  are components of the joint-type  $T(S) = \{t_v, v \in \mathbb{F}_2^L\}$ . To check (39) simply observe that if one arranges  $L$  codewords of  $S$  in an  $L \times n$  matrix and also adds  $j$  rows of zeros, then computation of  $h_j(S)$  can be done per-column: each column of weight  $w$  contributes

$$\min(w, L + j - w) = w - |2w - L - j|^+$$

to the sum. In view of expression (39) we will abuse notation and write

$$h_j(T(S)) \triangleq h_j(S).$$

We now observe that for symmetric codes satisfying (29) average-radii  $h_j(S)$  in fact determine the regular radius:

**Lemma 6.** *Consider an arbitrary code  $\mathcal{C}$  satisfying conclusion (29) of Lemma 5. Then for any subset  $S = \{c_1, \dots, c_L\} \subset \mathcal{C}$  we have*

$$\left| \text{rad}(0, c_1, \dots, c_L) - n \cdot \max_j h_j(\bar{T}_L(\mathcal{C})) \right| \leq 2^L(1 + \delta n), \quad (41)$$

where  $j$  in maximization (41) ranges over  $\{0, 1, 3, \dots, 2k + 1, \dots, L\}$  if  $L$  is odd and over  $\{0, 2, \dots, 2k, \dots, L\}$  if  $L$  is even.

*Proof:* For joint-types of size  $L$  and all  $j \geq 0$  we clearly have (cf. expression (39))

$$|h_j(T_1) - h_j(T_2)| \leq 2^{L-1} \|T_1 - T_2\|, \quad \forall T_1, T_2 \in \mathcal{T}_L. \quad (42)$$

We also trivially have

$$\frac{1}{n} \text{rad}(0, c_1, \dots, c_L) \geq h_j(S) \quad \forall j \geq 0. \quad (43)$$

Thus from (29) and (42) we already get

$$\frac{1}{n} \text{rad}(0, c_1, \dots, c_L) \geq \max_j h_j(\bar{T}_L(\mathcal{C})) - 2^{L-1} \delta.$$

It remains to show

$$\frac{1}{n} \text{rad}(0, c_1, \dots, c_L) \leq \max_j h_j(\bar{T}_L(\mathcal{C})) + \delta + \frac{2^L}{n}. \quad (44)$$

This evidently requires constructing a good center  $y$  for the set  $\{0, c_1, \dots, c_L\}$ . To that end fix arbitrary numbers  $q = (q_0, \dots, q_L) \in [0, 1]^L$ . Next, for each  $v \in \mathbb{F}_2^L$  let  $E_v \subset [n]$  be all coordinates on which restriction of  $\{c_1, \dots, c_L\}$  equals  $v$ . On  $E_v$  put  $y$  to have a fraction  $q_{|v|}$  of ones and remaining set to zeros (rounding to integers arbitrarily). Proceed for all  $v \in \mathbb{F}_2^L$ . Call resulting vector  $y(q) \in \mathbb{F}_2^n$ .

Denote for convenience  $c_0 = 0$ . We clearly have

$$\text{rad}(c_0, c_1, \dots, c_L) \leq \min_q \max_p \sum_{i=0}^L p_i |c_i - y(q)|, \quad (45)$$

where  $p = (p_0, \dots, p_L)$  is a probability distribution.

Denote

$$T(S) = \{t_v, v \in \mathbb{F}_2^L\} \quad (46)$$

$$\bar{T}_L(\mathcal{C}) = \{\bar{t}_v, v \in \mathbb{F}_2^L\} \quad (47)$$

We proceed to computing  $|c_i - y(q)|$ .

$$|c_i - y(q)| \leq n \sum_{v \in \mathbb{F}_2^L} t_v(q_{|v|} 1\{v(i) = 0\} + (1 - q_{|v|}) 1\{v(i) = 1\}) + 2^L, \quad (48)$$

where  $2^L$  comes upper-bounding the integer rounding issues and we abuse notation slightly by setting  $v(0) = 0$  for all  $v$  (recall that  $v(i)$  is the  $i$ -th coordinate of  $v \in \mathbb{F}_2^L$ ).

By (29) we may replace  $t_v$  with  $\bar{t}_v$  at the expense of introducing  $2^L \delta n$  error, so we have:

$$|c_i - y(q)| \leq n \sum_{v \in \mathbb{F}_2^L} \bar{t}_v(q_{|v|} 1\{v(i) = 0\} + (1 - q_{|v|}) 1\{v(i) = 1\}) + 2^L(1 + \delta n). \quad (49)$$

Next notice that the sum over  $v$  only depends on whether  $i = 0$  or  $i \neq 0$  (by symmetry of  $\bar{t}_v$ ). Furthermore, for any given weight  $w$  and  $i \neq 0$  we have

$$\sum_{v: |v|=w} 1\{v(i) = 1\} = \binom{L}{w} \frac{w}{L}.$$

Thus, introducing the random variable  $\bar{W}$ , cf. (39),

$$\mathbb{P}[\bar{W} = w] \triangleq \sum_{v: |v|=w} \bar{t}_v,$$

we can rewrite:

$$\begin{aligned} \sum_{v \in \mathbb{F}_2^L} \bar{t}_v(q_{|v|} 1\{v(i) = 0\} + (1 - q_{|v|}) 1\{v(i) = 1\}) \\ = \frac{1}{L} \mathbb{E}[\bar{W} + (L - 2\bar{W})q_{\bar{W}}]. \end{aligned} \quad (50)$$

For  $i = 0$  the expression is even simpler:

$$\sum_{v \in \mathbb{F}_2^L} \bar{t}_v(q_{|v|} 1\{v(0) = 0\} + (1 - q_{|v|}) 1\{v(0) = 1\}) = \mathbb{E}[q_{\bar{W}}].$$

Substituting derived upper bound on  $|c_i - y(q)|$  into (45) we can see that without loss of generality we may assume  $p_1 = \dots = p_L$ , so our upper bound (modulo  $O(\delta)$  terms) becomes:

$$\begin{aligned} \min_q \max_{p_1 \in [0, L^{-1}]} (1 - Lp_1) \mathbb{E}[q_{\bar{W}}] + p_1 \mathbb{E}[\bar{W} + (L - 2\bar{W})q_{\bar{W}}] \\ = \min_q \max_{p_1 \in [0, L^{-1}]} p_1 \mathbb{E}[\bar{W}] + \mathbb{E}[q_{\bar{W}}(1 - 2\bar{W}p_1)] \end{aligned}$$

By von Neumann's minimax theorem we may interchange min and max, thus continuing as follows:

$$= \max_{p_1 \in [0, L^{-1}]} \min_q p_1 \mathbb{E}[\bar{W}] + \mathbb{E}[q_{\bar{W}}(1 - 2\bar{W}p_1)] \quad (51)$$

$$= \max_{p_1 \in [0, L^{-1}]} p_1 \mathbb{E}[\bar{W}] - \mathbb{E}[|2\bar{W}p_1 - 1|^+]. \quad (52)$$

The optimized function of  $p_1$  is piecewise-linear, so optimization can be reduced to comparing values at slope-discontinuities and boundaries. The point  $p_1 = 0$  is easily excluded, while the rest of the points are given by  $p_1 = \frac{1}{L+j}$

with  $j$  ranging over the set specified in the statement of Lemma<sup>4</sup>. So we continue (52) getting

$$= \max_j \frac{1}{L+j} (\mathbb{E}[\bar{W}] - \mathbb{E}[|2\bar{W} - L - j|^+]) \quad (53)$$

We can see that expression under maximization is exactly  $h_j(\bar{T}_L(\mathcal{C}))$  and hence (44) is proved. ■

**Lemma 7.** *There exist constants  $C_1, C_2$  depending only on  $L$  such that for any  $\mathcal{C} \subset \mathbb{F}_2^n$  the joint-type  $\bar{T}_L(\mathcal{C})$  is approximately a mixture of product Bernoulli distributions<sup>5</sup>, namely:*

$$\left\| \bar{T}_L(\mathcal{C}) - \frac{1}{n} \sum_{i=1}^n \text{Bern}^{\otimes L}(\lambda_i) \right\| \leq \frac{C_1}{|\mathcal{C}|}, \quad (54)$$

where  $\lambda_i = \frac{1}{|\mathcal{C}|} \sum_{c \in \mathcal{C}} 1\{c(i) = 1\}$  be the density of ones in the  $i$ -th column of a  $|\mathcal{C}| \times n$  matrix representing the code. In particular,

$$\left| h_j(\bar{T}_L(\mathcal{C})) - \frac{1}{n} \sum_j g_j(\lambda_j) \right| \leq \frac{C_2}{|\mathcal{C}|}, \quad (55)$$

where functions  $g_j$  were defined in (16).

*Proof:* Second statement (55) follows from the first via (42) and linearity of  $h_j(T)$  in the type  $T$ , cf. (39). To show the first statement, let  $M = |\mathcal{C}|$ ,  $M_i = \lambda_i M$  and  $p_w$  – total probability assigned to vectors  $v$  of weight  $w$  by  $\bar{T}_L(\mathcal{C})$ . Then by computing  $p_w$  over columns of  $M \times n$  matrix we obtain

$$p_w = \frac{1}{n} \sum_{i=1}^n \frac{\binom{M_i}{w} \binom{M-M_i}{L-w}}{\binom{M}{L}}.$$

By a standard estimate we have for all  $w = \{0, \dots, L\}$ :

$$\frac{\binom{M_i}{w} \binom{M-M_i}{L-w}}{\binom{M}{L}} = \binom{L}{w} \lambda_i^w (1 - \lambda_i)^{L-w} + O\left(\frac{1}{M}\right),$$

with  $O(\cdot)$  term uniform in  $w$  and  $\lambda_i$ . By symmetry of the type  $\bar{T}_L(\mathcal{C})$  the result (54) follows. ■

**Lemma 8.** *Functions  $g_j$  defined in (16) are concave on  $[0, 1]$ .*

*Proof:* Let  $W_\lambda \sim \text{Bino}(L, \lambda)$  and  $V_\lambda \sim \text{Bino}(L-1, \lambda)$ . Denote for convenience  $\bar{\lambda} = 1 - \lambda$  and take  $j_0$  to be an integer

<sup>4</sup>The difference between odd and even  $L$  occurs due to the boundary point  $p_1 = \frac{1}{L}$  not being a slope-discontinuity when  $L$  is odd, so we needed to add it separately.

<sup>5</sup>Distribution  $\text{Bern}^{\otimes L}(\lambda)$  assigns probability  $\lambda^{|v|} (1-\lambda)^{L-|v|}$  to element  $v \in \mathbb{F}_2^L$ .

between 0 and  $L$ . We have then

$$\begin{aligned} & \frac{\partial}{\partial \lambda} \mathbb{E}[|W_\lambda - j_0|^+] \\ &= \sum_{w=j_0+1}^L \binom{L}{w} (w - j_0) \lambda^w \bar{\lambda}^{L-w} \{w\lambda^{-1} - (L-w)\bar{\lambda}^{-1}\} \end{aligned} \quad (56)$$

$$\begin{aligned} &= \binom{L}{j_0+1} (j_0+1) \lambda^{j_0} \bar{\lambda}^{L-j_0-1} \\ &+ \sum_{w=j_0+1}^{L-1} \left[ \binom{L}{w+1} (w+1-j_0)(w+1) \right. \\ &\quad \left. - \binom{L}{w} (w-j_0)(L-w) \right] \lambda^w \bar{\lambda}^{L-w-1} \end{aligned} \quad (57)$$

$$= L \binom{L-1}{j_0} \lambda^{j_0} \bar{\lambda}^{L-1-j_0} + L \sum_{w=j_0+1}^{L-1} \binom{L-1}{w} \lambda^w \bar{\lambda}^{L-1-w} \quad (58)$$

$$= L \mathbb{P}[V_\lambda \geq j_0], \quad (59)$$

where in (57) we shifted the summation by one for the first term under the sum in (56), and in (58) applied identities  $\binom{L}{w+1} = \binom{L}{w} \frac{L-w}{w+1} = \binom{L-1}{w} \frac{L}{w+1}$ . Similarly, if  $\theta \in [0, 1]$  we have

$$\frac{\partial}{\partial \lambda} \mathbb{E}[|W_\lambda - j_0 - \theta|^+] = L \mathbb{P}[V_\lambda \geq j_0+1] + L(1-\theta) \mathbb{P}[V_\lambda = j_0]. \quad (60)$$

Similarly, one shows (we will need it later in Lemma 9):

$$\frac{\partial}{\partial \lambda} \mathbb{P}[W_\lambda \geq j_0] = L \mathbb{P}[V_\lambda = j_0 - 1]. \quad (61)$$

Since clearly the function in (60) is strictly increasing in  $\lambda$  for any  $j_0$  and  $\theta$  we conclude that

$$\lambda \mapsto \mathbb{E}[|W_\lambda - j_0 - \theta|^+]$$

is convex. This concludes the proof of concavity of  $g_j$ . ■

*Proof of Lemma 4:* Our plan is the following:

- 1) Apply Elias-Bassalygo reduction to pass from  $\mathcal{C}'_n$  to a subcode  $\mathcal{C}''_n$  on an intersection of two spheres  $S_{\xi_0 n}$  and  $y + S_{\xi_1 n}$ .
- 2) Use Lemma 5 to pass to a symmetric subcode  $\mathcal{C}'''_n \subset \mathcal{C}''_n$ .
- 3) Use Lemmas 7-8 to estimate maxima of average radii  $h_j$  over  $\mathcal{C}'''_n$ .
- 4) Use Lemma 6 to transport statement about  $h_j$  to a statement on  $\tau_L(\mathcal{C}'''_n)$ .

We proceed to details. It is sufficient to show that for some constant  $C = C(L)$  and arbitrary  $\delta > 0$  estimate (24) holds with  $\epsilon_n = C\delta$  whenever  $n \geq n_0(\delta)$ . So we fix  $\delta > 0$  and consider a code  $\mathcal{C}' \subset S_{\xi_0 n} \subset \mathbb{F}_2^n$  with  $|\mathcal{C}'| \geq \exp\{nR' + o(n)\}$ . Note that for any  $r$ , even  $m$  with  $m/2 \leq \min(r, n-r)$  and arbitrary  $y \in S_r^n$  intersection  $\{y + S_m^n\} \cap S_r^n$  is isometric to the product of two lower-dimensional spheres:

$$\{y + S_m^n\} \cap S_r^n \cong S_{r-m/2}^r \times S_{m/2}^{n-r}. \quad (62)$$

Therefore, we have for  $r = \xi_0 n$  and valid  $m$ :

$$\sum_{y \in S_r^n} |\{y + S_m^n\} \cap \mathcal{C}'| = |\mathcal{C}'| \binom{\xi_0 n}{\xi_0 n - m/2} \binom{n(1-\xi_0)}{m/2}.$$

Consequently, we can select  $m = \xi_1 n - o(n)$ , where  $\xi_1$  defined in (27), so that for some  $y \in S_r^n$ :

$$|\{y + S_{\rho n}^n\} \cap \mathcal{C}'| > n.$$

Note that we focus on solution of (27) satisfying  $\xi_1 < 2\xi_0(1-\xi_0)$ . For some choices of  $R, \delta$  and  $\xi_0$  choosing  $\xi_1 > 2\xi_0(1-\xi_0)$  is also possible, but such a choice appears to result in a weaker bound.

Next, we let  $\mathcal{C}'' = \{y + S_{\rho n}^n\} \cap \mathcal{C}'$ . For sufficiently large  $n$  the code  $\mathcal{C}''$  will satisfy assumptions of Lemma 5 with  $K \geq \frac{1}{\delta}$ . Denote the resulting large symmetric subcode  $\mathcal{C}'''$ .

Note that because of (62) column-densities  $\lambda_i$ 's of  $\mathcal{C}'''$ , defined in Lemma 7, satisfy (after possibly reordering coordinates):

$$\sum_{i=1}^{\xi_0 n} \lambda_i = \xi_1 n/2 + o(n), \quad \sum_{i > \xi_0 n} \lambda_i = \xi_1 n/2 + o(n).$$

Therefore, from Lemmas 7-8 we have

$$\begin{aligned} h_j(\bar{T}_L(\mathcal{C}''')) &\leq \xi_0 g_j \left(1 - \frac{\xi_1}{2\xi_0}\right) \\ &+ (1 - \xi_0) g_j \left(\frac{\xi_1}{2(1-\xi_0)}\right) + \epsilon'_n + \frac{C_1}{|\mathcal{C}'''|}, \end{aligned} \quad (63)$$

where  $\epsilon'_n \rightarrow 0$ . Note that by construction the last term in (63) is  $O(\delta)$ . Also note that the first two terms in (63) equal  $\theta_j$  defined in (25).

Finally, by Lemma 6 we get that for any codewords  $c_1, \dots, c_L \in \mathcal{C}'''$ , some constant  $C$  and some sequence  $\epsilon''_n \rightarrow 0$  the following holds:

$$\frac{1}{n} \text{rad}(0, c_1, \dots, c_L) \leq \theta(\xi_0, R', L) + \epsilon''_n + C\delta.$$

By the initial remark, this concludes the proof of Lemma 4. ■

### C. Proof of Corollary 3

**Lemma 9.** For any odd  $L = 2a + 1$  there exists a neighborhood of  $x = \frac{1}{2}$  such that

$$\max_j g_j(x) = g_1(x), \quad (64)$$

maximum taken over  $j$  equal all the odd numbers not exceeding  $L$  and  $j = 0$ . We also have for some  $c > 0$

$$g_1(x) = \frac{1}{2} - 2^{-L-1} \binom{L}{\frac{L-1}{2}} + cx + O((2x-1)^2), \quad x \rightarrow \frac{1}{2}. \quad (65)$$

*Proof:* First, the value  $g_1(1/2)$  is computed trivially. Then from (60) we have

$$\frac{d}{dx} g_j(x) = \frac{L}{L+j} \left(1 - 2\mathbb{P}\left[V_x \geq \frac{L+j}{2}\right]\right), \quad (66)$$

where  $j \geq 1$  and  $V_x \sim \text{Bino}(x, L-1)$ . This implies (65). For future reference we note that (69) (below) and (61) imply

$$\begin{aligned} \frac{d}{dx} g_0(x) &= 1 - 2\mathbb{P}[V_x \geq \frac{L+1}{2}] - \mathbb{P}[V_x = \frac{L-1}{2}], \\ &V_x \sim \text{Bino}(x, L-1). \end{aligned} \quad (67)$$

By continuity, (64) follows from showing

$$g_1(1/2) > \max_{j \in \{0,3,5,\dots,L\}} g_j(1/2). \quad (68)$$

Next, consider  $W_x \sim \text{Bino}(x, L)$  and notice the upper-bound

$$g_j(x) \leq \frac{1}{L+j} \mathbb{E} [W_x 1\{W_x \leq a\} + (L+j-W_x) 1\{W_x \geq a+1\}].$$

Then, substituting expression for  $g_1(x)$  we get

$$g_1(x) - g_0(x) = \frac{1}{L} (\mathbb{P}[W_x \geq a+1] - g_1(x)) \quad (69)$$

$$g_1(x) - g_j(x) \geq \frac{j-1}{L+j} (g_1(x) - \mathbb{P}[W_x > a+1]). \quad (70)$$

Thus, to show (68) it is sufficient to prove that for  $x = 1/2$  we have

$$\mathbb{P}[W_{\frac{1}{2}} > a+1] < g_1(1/2) < \mathbb{P}[W_{\frac{1}{2}} \geq a+1]. \quad (71)$$

The right-hand inequality is trivial since  $\mathbb{P}[W_{\frac{1}{2}} \geq a+1] = 1/2$  while from (65) we know  $g_1(1/2) < 1/2$ . The left-hand inequality, after simple algebra, reduces to showing

$$\sum_{u=0}^{a-1} (2a+1-2u) \binom{2a+1}{u} < (2a+1) \binom{2a+1}{a}. \quad (72)$$

Notice, that

$$(n-2u) \binom{n}{u} = n \left[ \binom{n-1}{u} - \binom{n-1}{u-1} \right] \forall u \geq 0$$

and therefore

$$\sum_{u \leq \ell} (n-2u) \binom{n}{u} = n \binom{n-1}{\ell}.$$

Plugging this identity into the right-hand side of (72) we get

$$\begin{aligned} \sum_{u=0}^{a-1} (2a+1-2u) \binom{2a+1}{u} &= (2a+1) \binom{2a}{a-1} \\ &< (2a+1) \binom{2a}{a} < (2a+1) \binom{2a+1}{a} \end{aligned} \quad (73)$$

completing the proof of (72).  $\blacksquare$

*Proof of Corollary 3:* We first show that (20) implies (21). To that end, fix a small  $\epsilon > 0$  so that  $\frac{1}{2} - \epsilon$  belongs to the neighborhood existence of which is claimed in Lemma 9. Choose rate so that  $\delta_{LP1}(R) = 1/2 - \epsilon$  and notice that this implies

$$R = h(\epsilon^2 + o(\epsilon^2)), \quad (74)$$

By Lemma 9, the right-hand side of (20) is

$$\tau_L^*(0) - \text{const} \cdot \epsilon + o(\epsilon),$$

which together with (74) implies (21).

To prove (20) we use Theorem 1 with  $\delta = \delta_{LP1}(R)$ . Next, use concavity of  $g_j$ 's (Lemma 8) to relax (13) to

$$\limsup_{n \rightarrow \infty} \tau_L(\mathcal{C}_n) \leq \max_{j, \xi_0} g_j(\xi_0).$$

From (66) and (67) it is clear that  $\xi_0 \mapsto g_j(\xi_0)$  is monotonically increasing for all  $j \geq 0$  on the interval  $[0, 1/2]$ . Thus, we further have

$$\limsup_{n \rightarrow \infty} \tau_L(\mathcal{C}_n) \leq \max_j g_j(\delta_{LP1}(R)). \quad (75)$$

Bound (75) is valid for all  $R \in [0, 1]$  and arbitrary (odd/even  $L$ ). However, when  $R$  is small (say,  $R < R_0$ ) and  $L$  is odd,  $\delta_{LP1}(R)$  belongs to the neighborhood of  $1/2$  in Lemma 9 and thus (20) follows from (75) and (64).  $\blacksquare$

**Remark 2.** It is, perhaps, instructive to explain why Corollary 3 cannot be shown for even  $L$  (via Theorem 1). For even  $L$  the maximum over  $j$  of  $g_j(1/2 - \epsilon)$  is attained at  $j = 0$  and

$$g_0\left(\frac{1}{2} - \epsilon\right) = \tau_L^*(0) + c\epsilon^2 + O(\epsilon^3), \epsilon \rightarrow 0 \quad (76)$$

Therefore, for  $\delta_{LP1}(R) = \frac{1}{2} - \epsilon$  we get from (76) that the right-hand side of (75) evaluates to

$$\tau_L^*(0) - \text{const} \cdot \epsilon^2 \log \frac{1}{\epsilon}. \quad (77)$$

Thus, comparing (77) with (74) we conclude that for even  $L$  our bound on  $R_L^*(\tau)$  has negative slope at zero rate. Note that Blinovsky's bound (10) has negative slope at zero rate for both odd and even  $L$ .

#### D. Proof of Corollary 2

*Proof:* Instead of working with parameter  $\delta$  we introduce  $\beta \in [0, 1/2]$  such that

$$\delta = \frac{1}{2} - \sqrt{\beta(1-\beta)}.$$

We then apply Theorem 1 with  $h(\beta) = R$ . Notice that the bound on  $\xi_0$  in (14) becomes

$$0 \leq \xi_0 \leq \delta.$$

By a simple substitution  $\omega = \sqrt{\frac{\beta}{1-\beta}}$  we get from (11)

$$E_\beta(\delta) = \frac{1}{2} (\log 2 - h(\delta) + h(\beta)).$$

Therefore, when  $\xi_0 = \delta$  we notice that

$$R + h(\beta) - 2E_\beta(\xi_0) = R - \log 2 + h(\delta)$$

implying that defining equation for  $\xi_1$ , i.e. (15), coincides with (19).

Next for  $L = 3$  we compute

$$g_0(\nu) = \nu(1-\nu), \quad (78)$$

$$g_1(\nu) = \frac{3}{4}\nu - \frac{1}{2}\nu^3, \quad (79)$$

$$g_3(\nu) = \frac{1}{2}\nu. \quad (80)$$

Note that the right-hand side of (17) is precisely equal to

$$\delta g_1 \left(1 - \frac{\xi_1}{2\delta}\right) + (1-\delta) g_1 \left(\frac{\xi_1}{2(1-\delta)}\right).$$



So this corollary simply states that for  $L = 3$  the maximum in (13) is achieved at  $j = 1, \xi_0 = \delta$ . Let us restate this last statement rigorously: The maximum

$$\max_{j \in \{0,1,3\}} \max_{\xi_0 \in \delta} \xi_0 g_j \left(1 - \frac{x}{2\xi_0}\right) + (1 - \xi_0) g_j \left(\frac{x}{2(1 - \xi_0)}\right) \quad (81)$$

is achieved at  $j = 1, \xi_0 = \delta$ . Here  $x = x(\xi_0, \beta)$  is a solution of

$$2(h(\beta) - E_\beta(\xi_0)) = h(\xi_0) - \xi_0 h\left(\frac{x}{2\xi_0}\right) - (1 - \xi_0) h\left(\frac{x}{2(1 - \xi_0)}\right). \quad (82)$$

For notational convenience we will denote the function under maximization in (81) by  $g_j(\xi_0, x)$ .

We proceed in two steps:

- First, we estimate the maximum over  $\xi_0$  for  $j = 0$  as follows:

$$\max_{\xi_0} g_0(\xi_0, x) \leq \frac{\log 2 - R}{4 \log 2} \cdot \left(1 - \frac{1 - \delta}{a_{max}(1 - a_{max})}\right) + (1 - \delta) g_0(a_{min}), \quad (83)$$

where  $a_{max}, a_{min} \leq \frac{1}{2}$  are given by

$$a_{max} = h^{-1}(\log 2 - R), \quad (84)$$

$$a_{min} = h^{-1}\left(\log 2 - \frac{R}{1 - \delta}\right). \quad (85)$$

- Second, we prove that for  $j = 1$  function

$$\xi_0 \mapsto g_j(\xi_0, x(\xi_0))$$

is monotonically increasing.

Once these two steps are shown, it is easy to verify (for example, numerically) that  $g_1(\delta, x(\delta))$  exceeds both  $\frac{1}{2}\delta$  (term corresponding to  $j = 3$  in (81)) and the right-hand side of (83) (term corresponding to  $j = 0$ ). Notice that this relation holds for all rates. Therefore, maximum in (81) is indeed attained at  $j = 1, \xi_0 = \delta$ .

One trick that will be common to both steps is the following. From the proof of Lemma 4 it is clear that the estimate (24) is monotonic in  $R'$ . Therefore, in equation (82) we may replace  $E_\beta(\xi)$  with any upper-bound of it. We will use the well-known upper-bound, which leads to binomial estimates of spectrum components [15, (46)]:

$$E_\beta(\xi_0) \leq \frac{1}{2}(\log 2 + h(\beta) - h(\xi_0)). \quad (86)$$

Furthermore, it can also be argued that maximum cannot be attained by  $\xi_0$  so small that

$$h(\beta) - \frac{1}{2}(\log 2 + h(\beta) - h(\xi_0)) < 0.$$

So from now on, we assume that

$$h^{-1}(\log 2 - h(\beta)) \leq \xi_0 \leq \delta,$$

and that  $x = x(\xi_0) \leq 2\xi_0(1 - \xi_0)$  is determined from the equation:

$$\log 2 - R = \xi_0 h\left(\frac{x}{2\xi_0}\right) + (1 - \xi_0) h\left(\frac{x}{2(1 - \xi_0)}\right) \quad (87)$$

(we remind  $R = h(\beta)$ ).

We proceed to demonstrating (83). For convenience, we introduce

$$a_1 \triangleq 1 - \frac{x}{2\xi_0}, \quad (88)$$

$$a_2 \triangleq \frac{x}{2 - 2\xi_0}. \quad (89)$$

By constraints on  $x$  it is easy to see that

$$0 \leq a_2 \leq \min(a_1, 1 - a_1).$$

Therefore, we have

$$\log 2 - R = \xi_0 h(a_1) + (1 - \xi_0) h(a_2) \geq h(a_2)$$

and thus  $a_2 \leq a_{max}$  defined in (84). Similarly, we have

$$\log 2 - R = \xi_0 h(a_1) + (1 - \xi_0) h(a_2) \leq \xi_0 \log 2 + (1 - \xi_0) h(a_2),$$

and since  $\xi_0 \leq \delta$  we get that  $a_2 \geq a_{min}$  defined in (85).

Next, notice that  $\frac{h(x)}{x(1-x)}$  is decreasing on  $(0, 1/2]$ . Thus, we have

$$h(a_1) \geq g_0(a_1) 4 \log 2 \quad (90)$$

$$\begin{aligned} h(a_2) &\geq h(a_{max}) \frac{g_0(a_2)}{g_0(a_{max})} \\ &= \frac{\log 2 - R}{a_{max}(1 - a_{max})} g_0(a_2) \triangleq c \cdot g_0(a_2), \end{aligned} \quad (91)$$

where in the last step we introduced  $c > 4 \log 2$  for convenience. Consequently, we get

$$\begin{aligned} \log 2 - R &= \xi_0 h(a_1) + (1 - \xi_0) h(a_2) \\ &\geq 4 \log 2 \cdot \xi_0 g_0(a_1) + (1 - \xi_0) c \cdot g_0(a_2) \end{aligned} \quad (92)$$

$$\geq 4 \log 2 \cdot \xi_0 g_0(a_1) + (1 - \xi_0) c \cdot g_0(a_2) \quad (93)$$

$$= 4 \log 2 \cdot g_0(\xi_0, x) + (1 - \xi_0)(c - 4 \log 2) \cdot g_0(a_2) \quad (94)$$

$$\geq 4 \log 2 \cdot g_0(\xi_0, x) + (1 - \delta)(c - 4 \log 2) \cdot g_0(a_{min}). \quad (95)$$

Rearranging terms yield (83).

We proceed to proving monotonicity of (82). The technique we will use is general (can be applied to  $L > 3$  and  $j > 1$ ), so we will avoid particulars of  $L = 3, j = 1$  case until the final step.

Notice that regardless of the function  $g(\nu)$  we have the equivalence:

$$\begin{aligned} \frac{d}{d\xi_0} \xi_0 g(a_1) + (1 - \xi_0) g(a_2) &\geq 0 \iff \\ \frac{1}{2} \frac{dx}{d\xi_0} (g'(a_2) - g'(a_1)) &\geq \int_{a_2}^{a_1} (1 - x)(-g''(x)) dx - g'(a_2), \end{aligned} \quad (96)$$

where we recall definition of  $a_1, a_2$  in (88)-(89). Differentiating (87) in  $\xi_0$  (and recalling that  $R$  is fixed, while  $x = x(\xi_0)$  is an implicit function of  $\xi_0$ ) we find

$$\frac{dx}{d\xi_0} = -2 \frac{\log \frac{1-a_2}{a_1}}{\log \frac{1-a_2}{a_2} \frac{a_1}{1-a_1}} < 0.$$

Next, one can notice that the map  $(\xi_0, x, R) \mapsto (a_1, a_2)$  is a bijection onto the region

$$\{(a_1, a_2) : 0 \leq a_1 \leq 1, 0 \leq a_2 \leq a_1(1 - a_1)\}. \quad (97)$$

With the inverse map given by

$$\xi_0 = \frac{a_2}{1 - a_1 + a_2}, x = \frac{2a_2^2}{1 - a_1 + a_2}, \\ R = \log 2 - \xi_0 h(a_1) - (1 - \xi_0)h(a_2).$$

Thus, verifying (96) can as well be done for all  $a_1, a_2$  inside the region (97). Substituting  $g = g_1$  into (96) we get that monotonicity in (82) is equivalent to a two-dimensional inequality:

$$-2 \log \frac{1 - a_2}{a_1} \cdot (a_1^2 - a_2^2) \\ \geq (2a_1^2 - \frac{4}{3}(a_1^3 - a_2^3) - 1) \log \frac{1 - a_2}{a - 2} \frac{a_1}{1 - a_1}. \quad (98)$$

It is possible to verify numerically that indeed (98) holds on the set (97). For example, one may first demonstrate that it is sufficient to restrict to  $a_2 = 0$  and then verify a corresponding inequality in  $a_1$  only. We omit mechanical details. ■

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